

# ON HARMONIC FUNCTIONS AND THE SCHWARZ LEMMA

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**ABSTRACT.** We study the Schwarz lemma for harmonic functions and prove sharp versions for the cases of real harmonic functions and the norm of harmonic mappings.

## 1. INTRODUCTION

We first recall that the hyperbolic metric  $d_h(z, w)$  of the unit disk  $\mathbf{U} := \{z : |z| < 1\}$  is defined by

$$\tanh \frac{d_h(z, w)}{2} = \frac{|z - w|}{|1 - z\bar{w}|}.$$

The classical Schwarz lemma says that an analytic function  $f$  of the unit disk into itself is a contraction in the hyperbolic metric, i.e. for  $z, w \in \mathbf{U}$

$$(1.1) \quad \frac{|f(z) - f(w)|}{|1 - f(z)\overline{f(w)}|} \leq \frac{|z - w|}{|1 - z\bar{w}|}.$$

Letting  $z \rightarrow w$  we get

$$(1.2) \quad |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

In the particular case  $f(0) = 0$  we obtain from (1.1) the following

**Lemma 1.3.** *Let  $f : \mathbf{U} \rightarrow \mathbf{U}$  be an analytic function with  $f(0) = 0$ . Then  $|f(z)| \leq |z|$ ,  $z \in \mathbf{U}$ .*

In this paper we will consider harmonic functions  $f : \mathbf{U} \rightarrow B_n := \{x : |x| < 1\} \subset \mathbf{R}^n$  i.e. functions whose coordinate functions satisfy the Laplace equation. In this case, the Cauchy-Riemann equations need not be satisfied. During the past decade, harmonic functions have been extensively studied and many results from the theory of analytic functions have been extended for them. For instance, the Schwarz lemma has the following counterpart in this context.

**Lemma 1.4.** ([3], [2, p. 77] and [7, Theorem 4.4.6]) *Let  $f : \mathbf{U} \rightarrow \mathbf{U}$  be a harmonic functions with  $f(0) = 0$ . Then  $|f(z)| \leq \frac{4}{\pi} \tan^{-1} |z|$  and this inequality is sharp for each point  $z \in \mathbf{U}$ .*

Recently, several authors have proved the following three extensions of the Schwarz lemma, which is the motivation for our study.

2000 *Mathematics Subject Classification.* Primary 31C05; Secondary 30A10.

*Key words and phrases.* Harmonic functions, Hyperbolic metric, Unit disk.

FILE: quasi2.TEX, PRINTED: 2010-12-24, 4.47

**Proposition 1.5.** [1, Theorem 6.26] *Let  $f$  be a real harmonic function of the unit disk into  $(-1, 1)$ . Then there hold the following sharp inequality*

$$|\nabla f(0)| \leq \frac{4}{\pi}.$$

**Proposition 1.6.** [5, Proposition 3.1.] *Let  $f$  be a  $K$ -q.c. harmonic mapping of the unit disk into itself. Then*

$$|\nabla f(z)| \leq K \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

The previous proposition has its counterpart for the class of quasiconformal hyperbolic harmonic mappings. See [8] for related results.

**Proposition 1.7.** [6, Theorem 1.1] *Let  $f$  be an analytic function of the unit disk into the unit ball  $B_n \subset \mathbf{C}^n$ . Then there holds the following sharp inequality*

$$|\nabla|f(z)|| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

The results of this paper are:

**Theorem 1.8.** *Let  $f$  be a real harmonic function of the unit disk into  $(-1, 1)$ . Then the following sharp inequality holds*

$$(1.9) \quad |\nabla f(z)| \leq \frac{4}{\pi} \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad |z| < 1.$$

**Theorem 1.10.** *If  $f$  is a harmonic mapping of the unit disk into the unit ball  $B_n \subset \mathbf{R}^n$ , then*

$$(1.11) \quad |\nabla|f(z)|| \leq \frac{4}{\pi} \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad |z| < 1.$$

*The inequality (1.11) is sharp for each  $z$ .*

**Theorem 1.12.** *Let  $f$  be a harmonic function of the unit disk into  $(-1, 1)$ . Then we have*

$$d_h(f(z), f(w)) \leq \frac{4}{\pi} d_h(z, w),$$

*where  $d_h$  is hyperbolic distance in  $(-1, 1)$  and  $\mathbf{U}$  respectively.*

**Remark 1.13.** *Theorem 1.8 is an improvement and an extension of Proposition 1.5. Notice that*

$$\frac{4}{\pi} (1 - |f(0)|^2) \leq \frac{4}{\pi}.$$

*Theorem 1.10 is a harmonic version of Proposition 1.7.*

The following example is taken from [4] but it is used there in some other context. It shows that we do not have any complex version of Theorem 1.8, without imposing quasiconformality (as is done in Proposition 1.6).

**Example 1.14.** Let  $z = re^{it}$  and let  $f$  be a harmonic diffeomorphism of the unit disk onto itself such that  $\partial_r f$  exists in the boundary point 1, and  $\partial_t f$  is not bounded on  $z = 1$ . Then

$$\overline{\lim}_{r \rightarrow 1} \frac{1 - |f(r)|^2}{1 - r^2} = |f(1)| \left| \frac{\partial f(r)}{\partial r} \right|_{r=1} = M < \infty$$

but

$$\sup_{r \in [0,1]} |\nabla f(r)| = \infty.$$

For instance: let  $\psi$  be the function defined on  $[-\pi, \pi]$  by

$$\psi(\theta) = a \left( \theta + \int_0^\theta -\log|x| dx \right),$$

where  $a$  is a positive constant such that

$$a \left( 2\pi + \int_{-\pi}^\pi -\log|x| dx \right) = 2\pi.$$

Take  $F(\theta) = e^{i\psi(\theta)}$  and define  $f(z) = P[F](z)$ .

## 2. PROOFS

*Proof of Theorem 1.8.* Let  $h$  be the harmonic conjugate of  $f$ . Then  $a = f + ih$  maps the unit disk  $\mathbf{U}$  into the vertical strip  $S = \{w : -1 < \Re w < 1\}$ .

Making use of the conformal map

$$g(z) = \frac{2i}{\pi} \log \frac{1+z}{1-z}$$

of the unit disk  $\mathbf{U}$  onto the strip  $S$  we see that there exists an analytic function  $b : \mathbf{U} \rightarrow \mathbf{U}$ , such that

$$a(z) = \frac{2i}{\pi} \log \frac{1+b(z)}{1-b(z)}.$$

By Schwarz lemma for analytic functions we have

$$|b'(z)| \leq \frac{1 - |b(z)|^2}{1 - |z|^2}.$$

On the other hand

$$a'(z) = \frac{4i}{\pi} \frac{b'(z)}{1 - b(z)^2}$$

and

$$|\nabla f| = |a'(z)|.$$

We will find the best possible constant  $C$  such that

$$|\nabla f(z)| \leq C \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

As

$$|a'(z)| \leq \frac{4}{\pi} \frac{1 - |b(z)|^2}{|1 - b(z)^2|} \frac{1}{1 - |z|^2}$$

it will be enough to find the best possible constant  $C$  such that

$$\frac{4}{\pi} \frac{1 - |b(z)|^2}{|1 - b(z)^2|} \frac{1}{1 - |z|^2} \leq C \frac{1 - |\Re a(z)|^2}{1 - |z|^2}$$

or what is the same

$$\frac{4\pi}{(\pi^2 - 4|\arg \frac{1+b}{1-b}|^2)} \frac{1-|b|^2}{|1-b^2|} \leq C, |b| < 1.$$

Let  $\omega = \frac{1+b}{1-b} = re^{it}$ . Then  $-\pi/2 \leq t \leq \pi/2$  and

$$1-|b|^2 = \frac{4r \cos t}{r^2 + 2r \cos t + 1},$$

$$|1-b^2| = \frac{4r|e^{it}|}{r^2 + 2r \cos t + 1},$$

and hence the last inequality with the constant  $C = 4/\pi$  follows from

$$\frac{|\cos t|}{1 - \frac{4}{\pi^2}t^2} \leq 1$$

which holds for  $t \in (-\pi/2, \pi/2)$ . This yields (1.9).

To show that the inequality (1.9) is sharp, take the harmonic function

$$f(z) = \frac{2}{\pi} \arctan \frac{2y}{1-x^2-y^2}.$$

It is easy to see that

$$|\nabla f(0)| = \frac{4}{\pi} = \frac{4}{\pi} \frac{1-|f(0)|^2}{1-0^2}.$$

□

*Proof of Theorem 1.10.* Consider the function  $g(z) = \langle f(z), b \rangle$ , where  $|b| = 1$ ,  $b \in \mathbf{R}^n$ . Then by Theorem 1.8, for  $|h| = 1$ ,  $h \in \mathbf{C}$  we have

$$(2.1) \quad \langle \nabla f(z)h, b \rangle \leq \frac{4}{\pi} \frac{1-|g(z)|^2}{1-|z|^2}.$$

Let

$$b = \frac{f(z)}{|f(z)|}$$

and

$$h = \frac{\nabla f(z)^t b}{|\nabla f(z)^t b|}.$$

Here  $\nabla f(z)^t$  is the transpose of the matrix  $\nabla f(z)$ . Then (2.1) coincides with

$$|\nabla f|| \leq \frac{4}{\pi} \frac{1-|f(z)|^2}{1-|z|^2}.$$

It follows the desired conclusion. □

*Proof of Theorem 1.12.* Let  $m$  be a Möbius transformation of the unit disk onto itself that maps the points 0 and  $r > 0$  to the points  $z$  and  $w$ . Then the function  $f \circ m$  is also harmonic. Denote it by  $f$  as well. Without loss of generality assume that  $f(z) \leq f(w)$ . Let  $C = \{t \in [0, r] : f'(t) > 0\}$ . Then  $C = \bigcup_{n=0}^{\infty} C_n$ , where  $C_n$  are some intervals in  $[0, r]$ . Let  $A_1 = C_1$ ,  $A_2 = C_2 \setminus \{t : f(t) \in f(A_1)\}$ , and by induction, let

$$A_{n+1} = \{t \in C_n : f(t) \notin f(\bigcup_{k=1}^n A_k)\}.$$

Take

$$A = \bigcup_{k=1}^{\infty} A_k.$$

Denote by  $|\cdot|$  the Lebesgue measure on real line. Then  $[f(z), f(w)] \subset f([0, r])$  and  $A$  is a measurable subset of  $[0, r]$  such that  $|[f(z), f(w)]| = |f(A)|$  and  $f$  is injective in  $A$ . Therefore by using (1.9) we have

$$\begin{aligned} d_h(f(z), f(w)) &= \int_{f(z)}^{f(w)} \frac{|d\omega|}{1 - |\omega|^2} \\ &= \int_A \frac{|df(t)|}{1 - |f(t)|^2} \\ &\leq \frac{4}{\pi} \int_A \frac{|dt|}{1 - |t|^2} \\ &\leq \frac{4}{\pi} \int_{[0, r]} \frac{|dt|}{1 - |t|^2} = \frac{4}{\pi} d_h(z, w) \end{aligned}$$

□

**Acknowledgment.** This work was completed during the visit of the first author to the Universities of Helsinki and Turku in October 2010. The first author was partially supported by Ministry of education of Montenegro. This research was supported by Academy of Finland, project 2600066611.

## REFERENCES

- [1] S. AXLER, P. BOURDON, W. RAMEY: *Harmonic function theory*, Springer Verlag New York 2000.
- [2] P. DUREN: *Harmonic mappings in the plane*, Cambridge University Press, 2004.
- [3] E. HEINZ: *On one-to-one harmonic mappings*, Pacific J. Math. **9** (1959), 101-105.
- [4] D. KALAJ: *Harmonic functions and harmonic quasiconformal mappings between convex domains*, Thesis, Beograd 2002.
- [5] M. KNEŽEVIĆ, M. MATELJEVIĆ: *On the quasi-isometries of harmonic quasiconformal mappings*, J. Math. Anal. Appl., **334**: 1 (2007), 404-413.
- [6] M. PAVLOVIĆ: *A Schwarz lemma for the modulus of a vector-valued analytic function*. to appear in Proc. Amer. Math. Soc. DOI: 10.1090/S0002-9939-2010-10578-6
- [7] M. PAVLOVIĆ: *Introduction to function spaces on the disk*. 20. Matematički Institut SANU, Belgrade, 2004. vi+184 pp.
- [8] T. WAN: *Constant mean curvature surface, harmonic maps, and universal Teichmüller space*, J. Differential Geom. **35**: 3 (1992), 643–657.

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